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Some three-color Ramsey numbers, $R(P_4, P_5, C_k)$ and $R(P_4, P_6, C_k)$ ☆

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ABSTRACT

For given graphs G_1, G_2, G_3, the three-color Ramsey number $R(G_1, G_2, G_3)$ is defined to be the least positive integer n such that every 3-coloring of the edges of complete graph K_n contains a monochromatic copy of G_i colored with i, for some $1 \le i \le 3$. In this paper, we prove that $R(P_4, P_5, C_3) = 11$, $R(P_4, P_5, C_4) = 7$, $R(P_4, P_5, C_5) = 11$, $R(P_4, P_5, C_7) = 11$, $R(P_4, P_5, C_k) = k + 2$ for $k \ge 23$; $R(P_4, P_6, C_4) = 8$, $R(P_4, P_6, C_3) = R(P_4, P_6, C_5) = R(P_4, P_6, C_7) = 13$, $R(P_4, P_6, C_k) = k + 3$ for $k \ge 18$.

1. Introduction

In this paper, we shall only consider graphs without multiple edges or loops. For a graph G, the set of vertices of G is denoted by $V(G)$, the set of edges of G is denoted by $E(G)$, the cardinality of $V(G)$ is denoted by $|V(G)|$, the cardinality of $E(G)$ is denoted by $|E(G)|$, the length of its longest cycle is denoted by $c(G)$, the maximum degree of G is denoted by $\Delta(G)$, the complementary graph of G is denoted by $\overline{G}$. A *path* on i vertices is denoted by P_i. A *cycle* on i vertices is denoted by C_i. A *complete graph* of order i is denoted by K_i. A complete graph K_i dropping m edges is denoted by $K_i - me$. A complete bipartite graph $K_{m,n}$ dropping k edges is denoted by $K_{m,n} - ke$. A *star* graph, denoted by S_i, is a bipartite graph of order i with one partite set consisting of a single vertex, i.e. $S_i \cong K_{1,i-1}$, and the vertex with maximum degree is called the *center* of the star graph. A *book* graph, denoted by B_i, has $i + 2$ vertices and is

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Table 1
Some values of three-color Ramsey numbers for paths and cycles in mixed cases

$R(P_3, P_4, C_3) = 7$ [1]	$R(P_3, P_4, C_4) = 7$ [1]	$R(P_3, P_4, C_5) = 7$ [7]
$R(P_4, P_4, C_3) = 9$ [1]	$R(P_4, P_4, C_4) = 7$ [1]	$R(P_4, P_4, C_5) = 9$ [7]
$R(P_4, P_4, C_6) = 8$ [7]	$R(P_3, C_3, C_3) = 11$ [4]	$R(P_3, C_3, C_4) = 8$ [1]
$R(P_3, C_3, C_5) = 9$ [7]	$R(P_3, C_3, C_6) = 11$ [7]	$R(P_3, C_4, C_4) = 8$ [1]
$R(P_3, C_4, C_5) = 8$ [7]	$R(P_3, C_4, C_6) = 8$ [7]	$R(P_3, C_5, C_5) = 9$ [7]
$R(P_3, C_5, C_6) = 11$ [7]	$R(P_3, C_6, C_6) = 9$ [7]	

the result of a single vertex being connected to every vertex of a star S_{i+1}. Let $V_1, V_2 \subseteq V(G)$; we use $E_G(V_1, V_2)$ to denote the set of the edges between V_1 and V_2 in G. The *independent number* of a graph G is denoted by $\alpha(G)$. Please refer to [3] for more notation of graph theory.

For given graphs G_1, G_2, G_3, the three-color Ramsey number $R(G_1, G_2, G_3)$ is defined to be the least positive integer n such that every 3-coloring of the edges of complete graph K_n contains a monochromatic copy of G_i colored with i, for some $1 \leq i \leq 3$. For a 3-coloring (say *red*, *green* and *blue*) of graph G, we denote by G^r (resp. G^g, G^b) the subgraph induced by the *red* (resp. *green*, *blue*) edges of G. If there exists a 2-coloring (say *red* and *blue*) of the edges of G such that G contains neither G_1 with *red* nor G_2 with *blue*, G is called (G_1, G_2) *colorable*. For a graph G, the Turán number $T(n, G)$ is defined to be the maximum number of edges in any n-vertex graph which does not contain G. By $T'(n, G)$ we denote the maximum number of edges in any n-vertex non-bipartite graph which does not contain G.

Recently, many multicolor Ramsey numbers were investigated. Some exact values of three-color Ramsey numbers for paths and cycles in mixed cases were summarized in Table 1.

T. Dzido et al. obtained the following general multicolor results for cycles and paths.

Theorem 1 ([8,7]). *$R(P_3, C_k, C_k) = R(C_k, C_k) = 2k - 1$ for odd k, $k \geq 5$;*
$R(P_3, P_4, C_k) = k + 1$ for $k \geq 6$;
$R(P_4, P_4, C_k) = k + 2$ for $k \geq 6$;
$R(P_3, P_5, C_k) = k + 1$ for $k \geq 8$.

In this paper, we determine values of some three-color Ramsey numbers as follows. $R(P_4, P_5, C_3) = 11$, $R(P_4, P_5, C_4) = 7$, $R(P_4, P_5, C_5) = 11$, $R(P_4, P_5, C_7) = 11$, $R(P_4, P_5, C_k) = k + 2$ for $k \geq 23$; $R(P_4, P_6, C_4) = 8$, $R(P_4, P_6, C_k) = 13$ for odd k and $3 \leq k \leq 7$, $R(P_4, P_6, C_k) = k + 3$ for $k \geq 18$.

2. Proofs of main results

Firstly, we recall a result about the Turán number, which was proved by Faudree and Schelp in 1975.

Theorem 2 ([10]). *If G is a graph with $|V(G)| = kt + r$, $0 \leq r < k$, containing no path on $k + 1$ vertices, then $|E(G)| \leq t\binom{k}{2} + \binom{r}{2}$ with equality if and only if G is either $(tK_k) \cup K_r$ or $((t - l - 1)K_k) \cup (K_{(k-1)/2} + \overline{K}_{(k+1)/2+\mathrm{ik}+r})$ for some l, $0 \leq l < t$, where k is odd, $t > 0$, and $r = (k \pm 1)/2$.*

By Theorem 2, it is easy to have the following corollary.

Corollary 1. *For all integer n, $n \geq 3$,*

$$T(n, P_4) = \begin{cases} n, & n = 0 \pmod 3 \\ n - 1, & n = 1, 2 \pmod 3 \end{cases}$$

$$T(n, P_5) = \begin{cases} 3n/2, & n = 0 \pmod 4 \\ 3n/2 - 2, & n = 2 \pmod 4 \\ (3n - 3)/2, & n = 1, 3 \pmod 4 \end{cases}$$

$$T(n, P_6) = \begin{cases} 2n, & n = 0 \pmod 5 \\ 2n - 2, & n = 1, 4 \pmod 5 \\ 2n - 3, & n = 2, 3 \pmod 5. \end{cases}$$

Table 2
The values of $T(n, C_4)$ for n up to 21

n	0	1	2	3	4	5	6	7	8	9	10
$T(n, C_4)$	0	0	1	3	4	6	7	9	11	13	16
n	11	12	13	14	15	16	17	18	19	20	21
$T(n, C_4)$	18	21	24	27	30	33	36	39	42	46	50

Theorem 3 *([11]).* $R(P_n, P_m) = n + \lfloor m/2 \rfloor - 1$ *for all* $n \geq m \geq 2$.

Theorem 4 *([5]). Let* G *be a graph of order* $2k + 1 \geq 3$. *If the size of* G *is at least* $k^2 + k + 1$ *or the size of* G *is* $k^2 + k$ *and* $G \ncong K_{k,k+1}$, *then* G *contains a triangle.*

By Corollary 1 and Theorem 4, we have

Corollary 2. $T(11, P_4) = 10$, $T(11, P_5) = 15$, $T(11, C_3) = 30$.

Lemma 1. *If* $n \geq 4$ *and* $m \geq 1$, *then* $R(P_4, P_n, C_{2m+1}) \geq 2n + 1$.

Proof. Partition the vertices of K_{2n} into two sets V_1, V_2 with $|V_1| = |V_2| = n$. Since $R(P_4, P_n) > n$, there is a 2-coloring (say *red* and *green*) of the edges of $G[V_1]$ (resp. $G[V_2]$) such that there is no P_4 with *red* edges and no P_n with *green* edges. The edges of $E_{K_{2n}}(V_1, V_2)$ are colored with *blue*. K_{2n} with this 3-coloring contains no P_4 with *red* edges, no P_n with *green* edges and no odd cycle with *blue* edges. Thus, $R(P_4, P_n, C_{2m+1}) \geq 2n + 1$. □

Theorem 5. $R(P_4, P_5, C_3) = 11$.

Proof. By Lemma 1, we have $R(P_4, P_5, C_3) \geq 11$. Now, we will show that $R(P_4, P_5, C_3) \leq 11$. Suppose to the contrary that there exists a 3-coloring (say *red, green* and *blue*) of K_{11} such that K_{11}^r contains no P_4, K_{11}^g contains no P_5 and K_{11}^b contains no C_3. Since K_{11} has 55 edges, and by Corollary 2, $T(11, P_4) + T(11, P_5) + T(11, C_3) = 55$, the numbers of *red, green* and *blue* edges are 11, 15, 30, respectively. By Theorem 4, the subgraph induced by *blue* is isomorphic to $K_{5,6}$, and one of the partition sets has six vertices, in which the color of the edges is *red* or *green*. Since $R(P_4, P_5) = 6$, there is a P_4 with *red* or P_5 with *green* in this partition set, a contradiction. □

Theorem 6. $R(P_4, P_5, C_4) = 7$.

Before proving Theorem 6, we show some already known results and establish some corollaries and lemmas. In 1989, Clapham et al. [6] obtained some small Turán numbers for C_4, which are shown in Table 2.

By Corollary 1, we have

Corollary 3. $T(7, P_4) = 6$, $T(7, P_5) = 9$.

Lemma 2. *If* $n \geq 3$, *then* $R(P_4, P_n, C_4) \geq n + 2$.

Proof. Let H be a 3-coloring of the edges of K_{n+1}, where $V(H) = \{1, 2, \ldots, n + 1\}$, the *red* edges of H are the edges incident to 1, the blue edges of H are the edges incident to 2 except the edge $(1, 2)$, the color of the other edges of H is green. Then H is a (P_4, P_n, C_4)-graph, which implies $R(P_4, P_n, C_4) \geq n + 2$. □

Lemma 3. *Let* G *be a complete bipartite graph* $K_{3,4}$ *with two partite sets* X *and* Y, *where* $|X| = 3$ *and* $|Y| = 4$. *If each edge of* G *is colored green or blue, then either* G^g *contains* P_5 *or* G^b *contains* C_4.

Proof. Suppose to the contrary that G^g does not contain P_5 and G^b does not contain C_4. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$.

Claim 1. G^g *contains no* C_4.

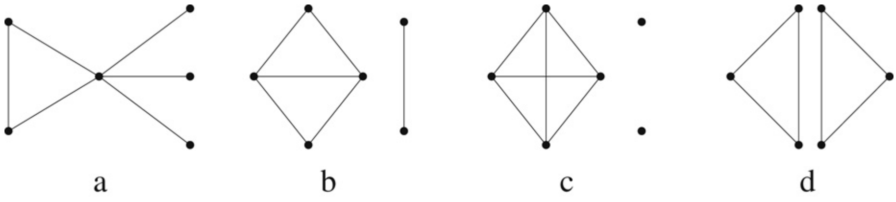

Fig. 1. P_5-free graphs of order 6 and size 6.

Otherwise, suppose $x_1y_2x_2y_1$ is a C_4 with *green* edges; then $x_1y_3, x_1y_4, x_2y_3, x_2y_4$ are colored with *blue* since G^g contains no P_5. Then $x_1y_3x_2y_4$ is a C_4 with *blue* edges, a contradiction.

Claim 2. *G^g contains no P_4.*

Otherwise, suppose $x_1y_1x_2y_2$ is a P_4 with *green* edges. Since by Claim 1, G^g contains no C_4, x_1y_2 is colored with *blue*. The edges x_1y_3, x_1y_4 and x_3y_2 are colored with *blue* since G^g contains no P_5. So, we have that x_3y_3 and x_3y_4 are colored with *green* since G^b contains no C_4. Then x_3y_1, x_2y_3 and x_2y_4 are colored with *blue* since G^g contains no P_5. Thus, $x_2y_4x_1y_3$ is a C_4 with *blue* edges, a contradiction.

Claim 3. *G^g contains no P_3.*

Otherwise, suppose there exists a P_3 with *green* edges in G. We prove Claim 3 via the following two cases.

Case 1. The two endpoints of P_3 are in X.

Suppose $P_3 = x_1y_1x_2$; then, by Claim 2, x_1y_2, x_1y_3, x_2y_2 and x_2y_3 are colored with *blue*. Thus, $x_1y_2x_2y_3$ is a C_4 with *blue* edges, a contradiction.

Case 2. The two endpoints of P_3 are in Y.

Suppose $P_3 = y_1x_1y_2$; then, by Claim 2, x_2y_1, x_2y_2, x_3y_1 and x_3y_2 are colored *blue*. Thus, $x_2y_1x_3y_2$ is a C_4 with *blue* edges, a contradiction.

Claim 4. *G^g contains no $2P_2$.*

Otherwise, suppose x_1y_1, x_2y_2 are two P_2 with *green* edges; then, by Claim 3, x_1y_3, x_1y_4, x_2y_3 and x_2y_4 are colored with *blue*. Thus $x_1y_3x_2y_4$ is a C_4 with *blue* edges, a contradiction.

By Claims 3 and 4, there is at most one edge colored with *green* in G; the other edges in G are all colored with *blue*. Thus, there exists a C_4 with *blue* edges in G, a contradiction, and the Lemma 3 holds. □

Lemma 4. *If G is a graph obtained by removing any two edges from K_6, and each edge of G is colored green and blue, then either G^g contains P_5 or G^b contains C_4.*

Proof. Suppose to the contrary that G contains neither a P_5 with *green* edges nor a C_4 with *blue* edges. By Table 2, $T(6, C_4) = 7$, so $|E(G^b)| \leq 7$. By Corollary 1, $T(6, P_5) = 7$, so $|E(G^g)| \leq 7$. Since $|E(G)| = 13$, $|E(G^g)| = 6$ and $|E(G^b)| = 7$ or $|E(G^g)| = 7$ and $|E(G^b)| = 6$.

Case 1. $|E(G^g)| = 6$ and $|E(G^b)| = 7$.

In this case, G^g is isomorphic to one of the following four graphs shown in Fig. 1, where all P_5-free graphs of order 6 and size 6 are listed. Since G is a graph obtained by removing any two edges from K_6, G^b is isomorphic to a graph obtained by removing any two edges of $\overline{G^g}$. If G^g is isomorphic to the graph shown in Fig. 1(a), then $\overline{G^g}$ contains a $K_5 - e$. If G^g is isomorphic to the graph shown in Fig. 1(b) or Fig. 1(c), then $\overline{G^g}$ contains a $K_{2,4}$. If G^g is isomorphic to the graph shown in Fig. 1(d), then $\overline{G^g}$ contains a $K_{3,3}$. It is easy to see that any graph obtained by removing two edges from $K_5 - e$ or $K_{3,3}$ or $K_{2,4}$ contains a C_4, a contradiction.

Case 2. $|E(G^g)| = 7$ and $|E(G^b)| = 6$.

In this case, by Theorem 2, $G^g \cong K_4 \cup K_2$. So G^b is isomorphic to a graph obtained by removing any two edges from $K_{2,4}$. Obviously, $K_{2,4} - 2e$ contains a C_4, a contradiction. □

Lemma 5. *If G is a graph of order 7 with no P_4, then either $\overline{G}$ contains a $K_6 - 2e$ or $V(G)$ can be partitioned into two sets X and Y such that $|X| = 3$, $|Y| = 4$ and $E_G(X, Y) = \phi$.*

Proof. By Corollary 1, $T(7, P_4) = 6$. We prove the Lemma 5 via the following five cases.

Case 1. $|E(G)| = 6$.

If G contains a C_3, say $C_3 = x_1x_2x_3$, let $X = \{x_1, x_2, x_3\}$, $Y = V(G) - X$; then $E_G(X, Y) = \phi$ since G contains no P_4. If G contains no C_3, then G is acyclic since G contains no P_4. Thus, G is a tree since $|E(G)| = |V(G)| - 1$ and G is acyclic. So $G \cong S_7$ since G is a tree and G contains no P_4. We have $\alpha(G) = 6$, then $\overline{G}$ contains a $K_6 - 2e$.

Case 2. $|E(G)| = 5$.

If G contains a C_3, we have $E_G(X, Y) = \phi$ (see case 1). If G contains no C_3, then G is a forest with two connected components, specifically, $G \cong S_6 \cup K_1$ or $G \cong S_5 \cup K_2$ or $G \cong S_4 \cup S_3$.

Subcase 1. $G \cong S_6 \cup K_1$.

In this subcase, $\alpha(G) = 6$; then $\overline{G}$ contains a $K_6 - 2e$.

Subcase 2. $G \cong S_5 \cup K_2$.

In this subcase, $\overline{G}$ also contains a $K_6 - 2e$.

Subcase 3. $G \cong S_4 \cup S_3$.

Let $X = S_3$ and $Y = S_4$; then $E_G(X, Y) = \phi$ since G contains no P_4.

Case 3. $|E(G)| = 4$.

If G contains a C_3, we have $E_G(X, Y) = \phi$ (see case 1). If G contains no C_3, then G is a forest with three connected components.

Subcase 1. $G \cong S_5 \cup 2K_1$.

In this subcase, $\alpha(G) = 6$; then $\overline{G}$ contains a $K_6 - 2e$.

Subcase 2. $G \cong S_4 \cup K_2 \cup K_1$ or $G \cong S_3 \cup K_3 \cup K_1$ or $G \cong S_3 \cup 2K_2$.

In this subcase, $\overline{G}$ contains a $K_6 - 2e$.

Case 4. $|E(G)| = 3$.

If G contains a C_3, we have $E_G(X, Y) = \phi$ (see case 1). If G contains no C_3, then G is a forest with four connected components.

Subcase 1. $G \cong S_4 \cup 3K_1$.

In this subcase, $\alpha(G) = 6$; then $\overline{G}$ contains a $K_6 - 2e$.

Subcase 2. $G \cong S_3 \cup K_2 \cup 2K_1$ or $G \cong 3K_2 \cup K_1$.

In this subcase, $\overline{G}$ contains a $K_6 - 2e$.

Case 5. $|E(G)| \leq 2$.

In this case, $\overline{G}$ contains a $K_6 - 2e$. □

Proof of Theorem 6. By Lemma 2, we have $R(P_4, P_5, C_4) \geq 7$. We will show that $R(P_4, P_5, C_4) \leq 7$ as follows. Suppose to the contrary that there exists a 3-coloring (say *red*, *green* and *blue*) of the edges of K_7 such that there is no P_4 with *red* edges, no P_5 with *green* edges and no C_4 with *blue* edges. By Lemma 5, $\overline{G^r}$ contains a $K_6 - 2e$ or $V(G^r)$ can be partitioned to two sets X and Y such that $|X| = 3$, $|Y| = 4$ and $E_{G^r}(X, Y) = \phi$.

Case 1. $\overline{G^r}$ contains a $K_6 - 2e$.

In this case, the edges of the subgraph $K_6 - 2e$ are colored with *green* or *blue*. By Lemma 4, G contains either a P_5 with *green* edges or a C_4 with *blue* edges, a contradiction.

Case 2. $V(G^r)$ can be partitioned to two sets X and Y such that $|X| = 3$, $|Y| = 4$ and $E_{G^r}(X, Y) = \phi$.

In this case, the edges between X and Y are colored with *green* or *blue*. By Lemma 3, G contains either a P_5 with *green* edges or a C_4 with *blue* edges, a contradiction. Therefore, in any case, Theorem 6 holds. □

Theorem 7. $R(P_4, P_5, C_5) = 11$.

In order to prove Theorem 7, we need the following theorems and corollaries.

Theorem 8 (*[9]*). *If $|E(G)| \geq |V(G)|$, then $c(G) > 2|E(G)|/|V(G)|$.*

Theorem 9 (*[2]*). *If G is a graph with $|E(G)| > |V(G)|^2/4$, then G contains C_k for every k, where $3 \leq k \leq c(G)$.*

By Theorems 8 and 9, we have

Corollary 4. *If G is a graph of order 11 and $|E(G)| > 30$, then G contains C_k for $3 \leq k \leq 6$ and $T(11, C_5) \leq 30$.*

Proof of Theorem 7. By Lemma 1, we have $R(P_4, P_5, C_5) \geq 11$. Now, we will show that $R(P_4, P_5, C_5) \leq 11$ as follows. Let $G = K_{11}$ and suppose to the contrary that there exists a 3-coloring (say *red*, *green* and *blue*) of the edges of G such that G^r contains no P_4, G^g contains no P_5 and G^b contains no C_5. By Corollaries 1 and 4, $T(11, P_4) = 10$, $T(11, P_5) = 15$ and $T(11, C_5) \leq 30$. Since the edge number of K_{11} is 55, $E(G^r) = 10$, $E(G^g) = 15$ and $E(G^b) = 30$. By Theorem 2, $E(G^g) = 15$ and G^g contains no P_5 if and only if $G^g \cong K_3 \cup 2K_4$; $E(G^r) = 10$ and G^r contains no P_4 if and only if $G^r \cong S_{11}$ or $G^r \cong K_3 \cup S_8$ or $G^r \cong 2K_3 \cup S_5$ or $G^r \cong 3K_3 \cup K_2$. So, $G - E(G^g) \cong K_{3,4,4}$, where $K_{3,4,4}$ is a complete three-partite graph with partite sets $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2, z_3, z_4\}$. We consider the following four cases.

Case 1. $G^r \cong S_{11}$.

Since $\Delta(K_{3,4,4}) = 8 < \Delta(S_{11}) = 10$, $G^r \ncong S_{11}$, a contradiction.

Case 2. $G^r \cong K_3 \cup S_8$.

Since $K_{3,4,4}$ is a complete three-partite graph, the vertices of K_3 in G^r are in different partite sets, say $K_3 = x_1y_1z_1$. Since G^r contains no P_4, any vertex $u \in V(G) - \{x_1, y_1, z_1\}$ is not adjacent to any vertex $v \in \{x_1, y_1, z_1\}$ in G^r. Thus, $\Delta(G^r) \leq 6$. Since $\Delta(S_8) = 7$, $G^r \ncong K_3 \cup S_8$, a contradiction.

Case 3. $G^r \cong 2K_3 \cup S_5$.

Without loss of generality, we suppose two K_3 in G^r are $x_1y_1z_1$ and $x_2y_2z_2$. We consider the following two subcases.

Subcase 1. The *center* of S_5 is in X.

In this subcase, x_3 is the *center* of S_5, and S_5 in G^r is formed by x_3, y_3, y_4, z_3, z_4. Thus, $x_1z_3y_1x_2y_3$ is a C_5 with *blue* edges, a contradiction.

Subcase 2. The *center* of S_5 is in Y or Z.

By symmetry of Y and Z, we suppose the *center* of S_5 with *red* edges is in Y, say y_3. Since y_3 is not adjacent to y_4 in G^r, then the degree of y_3 in G^r is less than 4, which implies it is not the *center* of S_5, a contradiction.

Case 4. $G^r \cong 3K_3 \cup K_2$.

Without loss of generality, we suppose three K_3 in G^r are $x_1y_1z_1$, $x_2y_2z_2$ and $x_3y_3z_3$; then x_4y_4 forms K_2. Thus, $x_3y_2z_3y_4z_1$ forms a C_5 with *blue* edges, a contradiction.

Therefore, Theorem 7 holds. □

Theorem 10. $R(P_4, P_5, C_7) = 11$.

The proof of Theorem 10 is similar to the proof of Theorem 7, so we omit the details, and just give the following Corollaries 5 and 6 that are useful in the proof of Theorem 10.

Theorem 11 (*[8]*). *Let $w(n, k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1)$, where $r = (n-1) \bmod (k-1)$; then for odd integers $k \geq 5$, $T'(n, C_k) = w(n, k-1)$, where $k \leq n \leq 2k-2$.*

By Theorem 11, we have

Corollary 5. $T'(11, C_7) = 30$.

Since the size of any bipartite graph of order 11 is at most 30 and any bipartite graph contain no odd cycle, then by Corollary 5, we have

Corollary 6. $T(11, C_7) = 30$.

Theorem 12. $R(P_4, P_5, C_k) = k + 2$ *for* $k \geq 23$.

Proof. Since $R(P_4, P_4, C_k) = k + 2$ for $k \geq 6$ [8], $R(P_4, P_5, C_k) \geq R(P_4, P_4, C_k) = k + 2$ for $k \geq 23$. So we only need to show that $R(P_4, P_5, C_k) \leq k + 2$ for $k \geq 23$. By Theorem 11 and Corollary 1, we have that $T'(k + 2, C_k) = \frac{1}{2}k^2 - \frac{3}{2}k + 7$ for $k \geq 5$, $T(k + 2, P_4) \leq k + 2$ and $T(k + 2, P_5) \leq \frac{3(k+2)}{2}$. Since $T'(k + 2, C_k)$ is greater than the maximal number of edges in a bipartite graph with $k + 2$ vertices, $T(k + 2, C_k) = T'(k + 2, C_k)$. We can see that $T(k + 2, P_4) + T(k + 2, P_5) + T(k + 2, C_k) \leq k + 2 + \frac{3(k+2)}{2} + \frac{1}{2}k^2 - \frac{3}{2}k + 7 < \binom{k+2}{2}$ for $k \geq 23$. So we have $R(P_4, P_5, C_k) \leq k + 2$. $\square$

Theorem 13. $R(P_4, P_6, C_4) = 8$.

Proof. By Lemma 2, we have $R(P_4, P_6, C_4) \geq 8$. By a method similar to that used in the proof of Theorem 7, we can prove $R(P_4, P_6, C_4) \leq 8$, so Theorem 13 holds. This result can also be proved by computer search, since $R(P_4, P_6, C_4)$ is small. The sketch of the computer search is as follows. (1) Generate 274 P_6-free graphs of order 8 with the program *nauty* [12]. (2) For the complement graph G of each P_6-free graph, test whether G is (P_4, C_4) *colorable*. Using a backtrack program with a pruning strategy, we find that each such graph is not (P_4, C_4)-*colorable*, which tells us the result stated in Theorem 13 holds. $\square$

Theorem 14. $R(P_4, P_6, C_k) = 13$, *for odd* k, $3 \leq k \leq 7$.

By Theorems 8 and 9, we have

Corollary 7. *If* G *is a graph of order* 13 *and* $|E(G)| > 42$, *then* G *contains* C_k, *and* $T(13, C_k) \leq 42$, *for* $3 \leq k \leq 7$.

Proof of Theorem 14. By Lemma 1, we have $R(P_4, P_6, C_k) \geq 13$ for odd k. Now, we will show that $R(P_4, P_6, C_k) \leq 13$. Let $G = K_{13}$ and suppose to the contrary that there exists a 3-coloring (say *red*, *green* and *blue*) of the edges of G such that G^r contains no P_4, G^g contains no P_6 and G^b contains no C_k. By Corollaries 1 and 7, we have $|E(G^r)| + |E(G^g)| + |E(G^b)| \leq T(13, P_4) + T(13, P_6) + T(13, C_k) \leq 77$. Since the edge number of K_{13} is 78, this is a contradiction. Therefore, Theorem 14 holds. $\square$

Theorem 15. $R(P_4, P_6, C_k) = k + 3$ *for* $k \geq 18$.

Proof. Let $G = (V, E)$ be a complete graph of order $k + 2$, where $V = \{v_1, v_2, \dots, v_{k+2}\}$, $E = \{v_iv_j | 1 \leq i \neq j \leq k + 2\}$. Define $V_1 = \{v_1, v_2, \dots, v_{k-1}\}$, $V_2 = \{v_k, v_{k+1}, v_{k+2}\}$, $E_1 = \{v_iv_j | 1 \leq i \neq j \leq k - 1\}$, $E_2 = \{v_kv_i | 1 \leq i \leq k - 1\}$, $E_3 = \{v_iv_j | k + 1 \leq i \leq k + 2, 1 \leq j \leq k - 1\}$ and $E_4 = \{v_kv_{k+1}, v_kv_{k+2}, v_{k+1}v_{k+2}\}$. The edges in E_1 and E_4 are colored with *blue*, the edges in E_2 are colored with *red* and the edges in E_3 are colored with *green*; then G contains neither P_4 with *red* edges, nor P_6 with *green* edges nor C_k with *blue* edges. Thus $R(P_4, P_6, C_k) \geq k + 3$.

Now, we only need to prove $R(P_4, P_6, C_k) \leq k + 3$. By Theorem 11 and Corollary 1, we have that $T'(k + 3, C_k) = \frac{1}{2}k^2 - \frac{3}{2}k + 11$ for $k \geq 5$, $T(k + 3, P_4) \leq k + 3$ and $T(k + 3, P_6) \leq 2(k + 3)$. Since $T'(k + 3, C_k)$ is greater than the maximal number of edges in a bipartite graph with $k + 3$ vertices for $k \geq 18$, $T(k + 3, C_k) = T'(k + 3, C_k)$ for $k \geq 18$. We can see that $T(k + 3, P_4) + T(k + 3, P_6) + T(k + 3, C_k) \leq k + 3 + 2(k + 3) + \frac{1}{2}k^2 - \frac{3}{2}k + 11 < \binom{k+3}{2}$ for $k \geq 18$. So we have $R(P_4, P_6, C_k) \leq k + 3$ for $k \geq 18$. Therefore, Theorem 15 holds. $\square$

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